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A matrix phase for the ϕ^4 scalar field on the fuzzy sphere

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Abstract: The critical properties of the real ϕ^4 scalar field theory are studied numerically on the fuzzy sphere. The fuzzy sphere is a finite matrix (non-commutative) approximation of the algebra of functions on the usual two dimensional sphere. It is also one of the simplest examples of a non-commutative space to study field theory on. Aside from the usual disordered and uniform phases present in the commutative scalar field theory, we find and discuss in detail a new phase called a matrix phase because the geometry of the fuzzy sphere, as expressed by the kinetic term, becomes negligible there. This highlights a new aspect of UV-IR mixing, the unusual behaviour which arises naturally when taking the commutative limit of a non commutative field theory.

Keywords: Nonperturbative Effects, Non-Commutative Geometry.

Contents

1. Introduction

The fuzzy approximation scheme [1, 2] consists in approximating the algebra of functions on a manifold with a finite dimensional algebra (i.e. a matrix algebra) instead of discretising the underlying space as a lattice approximation does.

Studying field theory on a non-commutative space such as the matrix algebras we are considering is covered by the framework of Connes' non-commutative geometry [3]. For reasons of simplicity, only the simpler case of the real scalar field will be considered in this paper. As a result, the only relevant differential operator is the laplacian and it is possible to consider a greatly simplified theory.

Indeed, approximating an algebra of functions as an algebra of matrices does not carry in itself any geometrical content, as is obvious from the fact that all fuzzy approximations must yield the same matrix algebras. What really defines the geometry is the choice of a derivation and scalar product. For the scalar product, we will normally arrange to use the canonical scalar product on the algebra of matrices

$$\langle\phi|\psi\rangle \propto \mathrm{Tr}(\phi^*\psi)\,. \tag{1.1}$$

As a consequence, complex conjugation will be associated with hermitian conjugation and thus, real fields will be approximated with hermitian matrices. Then, as far as the scalar field theory is concerned, the only required differential operator is the laplacian. The fuzzy approximation of the scalar field theory on a manifold will therefore be entirely determined by the choice of a laplacian and multiplicative coefficient in (1.1) for each matrix algebra.

However, even with such a simple scheme, only a few manifolds can be "fuzzified" in this way, including the complex projective planes $\mathbb{CP}^N$ [4] as well as their cartesian products [5]. Under certain conditions, it is also possible to approximate other spaces such as the spheres S^3 [6] and S^4 [7] by imbedding them in one of the fuzzy $\mathbb{CP}^N$. The non-commutative lattice [8] is another simple matrix space where numerical simulations have already been performed, although it is not a fuzzy space in the sense that it does not approximate a manifold but a lattice.

As a approximation scheme, this "fuzzification" is well suited to numerical simulations of field theories [9]. As a test run, the first fuzzy approximation to be investigated should be the simplest one, that of the two dimensional sphere $\mathbb{CP}^1 = S^2$. Besides, the two-dimensional plane can be viewed as the limit of a sphere of infinite radius.

In this paper, the fuzzy sphere and its properties will first be introduced in section 2. Then, the real scalar field theory on a sphere and on the fuzzy spheres will be presented in section 3. The following section 4 shows the results of the simulations. The phase boundaries between the three phases present and their scaling properties are derived. A careful analysis of this new phase is also presented in this section. Finally, the results are summarised and discussed in the Conclusion 5.

2. The fuzzy sphere

The simplest example of a fuzzy space is the fuzzy sphere [10]. As explained in the Introduction, for the purpose of studying a scalar field theory, the only ingredient required to fix the geometry is a laplacian operator and a scalar product on each matrix algebra. Since derivations on the commutative sphere can be viewed as infinitesimal $S\,\mathrm{SU}(2)$ transformations, the laplacian on a $(2s+1)\times(2s+1)$ matrix algebra, also denoted Mat_{2s+1}, can be guessed as

$$\mathcal{L}^2\phi = [L_i,[L_i,\phi]]\,, \tag{2.1}$$

where L_i are the angular momentum operators in the $2s+1$ dimensional irreducible representation of SU(2). The scalar product is chosen as proposed in (1.1), with a multiplicative coefficient such that the unit matrix has the same norm as the unit function on the sphere

$$\langle\phi|\psi\rangle = \frac{4\pi}{2s+1}\,\mathrm{Tr}(\phi^\dagger\psi)\,. \tag{2.2}$$

The spectrum of the proposed laplacian operator can be recognised from the adjoint action of angular momentum as

$$\mathcal{L}^2\hat{Y}_{lm} = l(l+1)\hat{Y}_{lm}\,, \qquad 0\le l\le 2s\,, \tag{2.3}$$

where the eigenfunctions $\hat{Y}_{lm}$ are the polarisation tensors whose normalisation is defined according to the chosen scalar product

$$\frac{4\pi}{2s+1}\,\mathrm{Tr}(\hat{Y}^\dagger_{lm}\hat{Y}_{lm}) = 1\,. \tag{2.4}$$

This is precisely the spectrum of the laplacian on the commutative sphere truncated at angular momentum $2s$, thus vindicating this choice.

A clean way of recognising the approximation of a sphere in these matrix algebras is to introduce a mapping which associates a function on the sphere with each matrix of the algebra Mat_{2s+1} and pulls back most of the structure on the algebra of functions of the sphere onto the matrix algebra [11]. There are various ways to define such a mapping, such as using coherent states [12]. However, the simplest one is given by

$$\mathcal{M}_s: \quad \mathrm{Mat}_{2s+1} \to \mathcal{C}^\infty(S^2) \tag{2.5}$$

$$M = \sum_{l=0}^{2s} \sum_{m=-l}^{l} c_{lm} \hat{Y}_{lm} \mapsto f(\mathbf{n}) = \sum_{l=0}^{2s} \sum_{m=-l}^{l} c_{lm} Y_{lm}(\mathbf{n}), \tag{2.6}$$

where the functions $Y_{lm}(\mathbf{n})$ are the usual spherical harmonics on the sphere, i.e. the eigenvectors of the laplacian operator on the sphere. By definition, this mapping $\mathcal{M}_s$ is linear and maps the laplacian $\mathcal{L}^2$ on Mat_{2s+1} onto the laplacian on the sphere. In fact, the three derivatives on the sphere $\nabla_l = i\varepsilon_{jkl} x_j \partial_k$ are pulled back to simple derivations on the matrix algebra given by

$$\mathcal{L}_i \phi = [L_i, \phi]. \tag{2.7}$$

By construction, the action of the group SU(2) is preserved on both sides. Furthermore, since the eigenvectors of the laplacian on the matrix space and on the sphere form orthonormal bases on their respective spaces, this mapping is an injective isometry. Its image, on which the mapping is one to one, $\mathcal{M}_s(\mathrm{Mat}_{2s+1})$ is given by all the functions with angular momentum only up to $2s$ and form a sequence of increasing (for the inclusion) sets which become dense in $\mathcal{C}^\infty(S^2)$ in the limit of infinite matrices. The matrix product is mapped to a (non-commutative) product of functions on the sphere called a $*$-product

$$\mathcal{M}_s(\phi\psi)(\mathbf{n}) = (\mathcal{M}_s(\phi) *_s \mathcal{M}_s(\psi))(\mathbf{n}), \tag{2.8}$$

which is evidently distinct from the usual (commutative) product of functions. It is possible to verify that in the limit of infinite matrices $s \to \infty$, the star product tends to the usual product. More precisely, for $(f_s, g_s) \in (\mathcal{M}_s(\mathrm{Mat}_{2s+1}))^2$ two functions with angular momentum truncated at $2s$, and $t \geq s$,

$$(f_s *_t g_s)(\mathbf{n}) = f_s(\mathbf{n}) g_s(\mathbf{n}) + \mathcal{O}\left(\frac{1}{t}\right). \tag{2.9}$$

Note in passing that complex conjugation of a function on the sphere pulls back to hermitian conjugation on the matrix algebra. Consequently, as proposed in the introduction, real functions pull back to hermitian matrices. Similarly, integration on the sphere which is similar to scalar product with the unit function pulls back to the trace on the matrix algebra.

Thus, in the limit when s goes to infinity, the mapping $\mathcal{M}_s$ becomes an isomorphism of algebras which preserves rotational invariance, the laplacian and the scalar product (2.2). This proves that the fuzzy spaces, as defined by the triple $(\mathrm{Mat}_{2s+1}, \mathcal{L}^2, \langle\cdot,\cdot\rangle)$ go over to the sphere in the limit of infinitely large matrices. Furthermore, we can deduce immediately the following approximation rule.

Approximation rule: given an algebraic expression on the sphere, it is possible to find a fuzzy approximation for it, which converges to it in the large matrix limit, by truncating all functions at momentum $2s$, replacing products by $*_s$-products everywhere and pulling back the expression into the matrix algebras with the mapping $\mathcal{M}_s$.

Another mapping with similar properties which is generally introduced is the one obtained by looking at the diagonal elements of a matrix in a coherent states representation. Compared to $\mathcal{M}_s$, this mapping trades the isometry property for the conservation of the notion of state, in the sense that it maps a state of Mat_{2s+1} into a state of $\mathcal{C}^\infty(S^2)$. In this case, the corresponding star product can also be expressed in a simple exact form [12].

This introduction to the fuzzy sphere described spheres of radius one. Getting spheres of different radius R, is just a matter of scaling the scalar product (1.1) and laplacian (2.1) appropriately:

$$\mathcal{L}^2 \to \frac{1}{R^2}\mathcal{L}^2\,, \tag{2.10}$$

$$\frac{4\pi}{2s+1}\,\mathrm{Tr}(\phi^\dagger\psi) \to \frac{4\pi R^2}{2s+1}\,\mathrm{Tr}(\phi^\dagger\psi)\,. \tag{2.11}$$

With the fuzzy sphere now defined and understood, we can move on to defining a real scalar field theory on it.

3. The real scalar field theory

The ϕ^4 scalar field theory on the two-dimensional sphere is given by the action[1]

$$S(\phi) = \int_{S^2} d^2\mathbf{n}(\phi\Delta\phi + r\phi^2 + \lambda\phi^4)\,, \tag{3.1}$$

with ϕ a real scalar field, $\Delta = -\nabla_i\nabla_i$ the laplacian on the sphere, r a mass parameter and λ an interaction constant. This particular model was chosen because it is simple and well studied. In fact, it is known that the diagrammatic expansion of this theory has only one divergent diagram, the tadpole diagram, is Borel resumable, and defines the field theory entirely.

Using the Approximation rule above, the action (3) for the real scalar field can be approximated by [13]

$$S(\phi) = \frac{4\pi}{2s+1}\,\mathrm{Tr}(\phi\mathcal{L}^2\phi + r\phi^2 + \lambda\phi^4)\,, \tag{3.2}$$

where ϕ must be an hermitian matrix. Again, it is possible to write a diagrammatic expansion for this theory [13]. There are more diagrams than in the continuum since the legs of the vertices do not commute anymore, although they can still be cyclically rotated. On the other hand, since the theory is defined on a finite dimensional algebra, all diagrams must be finite. Furthermore, since the action was obtained by the approximation rule, it is easy to

[1] A more usual form of this action can be obtained by changing the field ϕ to $\phi/\sqrt{2}$. Since the observables evaluated in this paper are typically homogeneous of degree two in the field, they are simply multiplied by an overall factor of two in this change of variable.

check that all the fuzzy diagrams are just approximations of their commutative counterparts obtained from the action (3). Thus, in the limit of infinite matrices, all finite diagrams converge to their commutative counterparts, while the tadpole diagrams must also diverge.

The approximation rule says nothing of the sub-dominant contribution to the tadpole diagrams however. In fact, the constant contribution of one of the two diagrams (the "non-planar" one where the exterior legs are not adjacent) does *not* converge to its usual continuum limit [13]. Thus, we see that the field theory described by the action (3.2) is *not* an approximation of the continuum field theory (3.1). This is what gives rise to the so called UV-IR mixing in the disordered phase, as studied in [13]. In the following, I will generically refer to "theories with UV-IR mixing" as such theories which do not converge to their commutative limits.

At this point, there are two interesting theories which can be studied. The first one described by the fuzzy action (3.2) with its associated UV-IR mixing is a simple example of a non-commutative theory such as those that have cropped up in particle theory recently. The other one is described by the same action (3.2) plus an additional damping term to ensure that the fuzzy theory does indeed go over to the commutative one [13]. Such a fuzzy theory will then be an alternative to the usual lattice discretisation of a scalar field theory.

As the first step toward a possible new approximation method of field theory, the latter theory holds the most potential. However, the former theory, which will be studied in this paper, proposes a non-perturbative analysis of a non-commutative field theory with UV-IR mixing, and more prosaically is simpler to implement. A similar study was done for this same model on the non-commutative lattice in [8].

Even though the scalar field on the commutative or fuzzy spheres cannot have a phase transition since both have finite volume, one may be found in the planar limit, i.e. in the limit where the matrices and the sphere become infinite $s, R \to \infty$. It is therefore convenient in the following to introduce explicitly the sphere radius R in the problem.

From eqs. (2.10), (2.11), it is clear that the real scalar field action with variable radius must take the form

$$S(\phi) = \frac{4\pi}{2s+1} \,\mathrm{Tr}(\phi \mathcal{L}^2 \phi + r R^2 \phi^2 + \lambda R^2 \phi^4)\,. \tag{3.3}$$

It will also be convenient later on to define the potential part of the action given by

$$V(\phi) = \frac{4\pi R^2}{2s+1} \,\mathrm{Tr}(r\phi^2 + \lambda \phi^4)\,. \tag{3.4}$$

Of the three parameters r, λ and R, only two are independent, and in the following, according to the situation, either the interaction parameter λ or the radius R will be set to one while the other parameter is allowed to vary.

4. The simulations

The theory simulated here is the one described by the action (3.3) with fields ϕ which are hermitian matrices. At least one critical line is expected to arise when the radius R goes to infinity, corresponding to the critical behaviour of the scalar field on the plane. Thus, the goal of the simulation will be to draw a phase diagram for this theory.

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The simplest way to test for a continuous phase transition is to look for peaks of the finite-volume susceptibility[2] given by

$$\chi = \left\langle \mathrm{Tr}(\phi)^2 \right\rangle - \langle | \mathrm{Tr}(\phi)| \rangle^2 \,, \tag{4.1}$$

where the expectation values are given by

$$\langle h(\phi) \rangle = \int \mathrm{d}\phi \frac{\mathrm{e}^{-S(\phi)}}{Z} h(\phi) \,, \tag{4.2}$$

with S the field action (3.3), h some algebraic expression in the field, and Z the partition function

$$Z = \int \mathrm{d}\phi \, \mathrm{e}^{-S(\phi)} \,. \tag{4.3}$$

For our purpose, the normalisation of the susceptibility is unimportant, contrary to the observables introduced next.

To get a better idea of the phase the field is in, it is also convenient to monitor a few other significant quantities. Using the mapping $\mathcal{M}$ introduced in (2.6), it is possible to associate functions on the sphere to matrices. Therefore, the coefficients c_{lm} in the expansion of a matrix on the basis of polarisation tensors (2.3)

$$\phi = \sum_{l,m} c_{lm} \hat{Y}_{lm} \,, \tag{4.4}$$

have an immediate classical interpretation. They will automatically average to zero though since they are linear, and thus odd, in the field while the action is even in it. For this reason, we chose the following quadratic even additional observables

$$\left\langle \|\phi\|^2 \right\rangle = \left\langle \sum_{l,m} |c_{lm}|^2 \right\rangle = \left\langle \frac{4\pi}{2s+1} \, \mathrm{Tr}(\phi^2) \right\rangle \tag{4.5}$$

$$\left\langle |c_{00}|^2 \right\rangle = \frac{4\pi}{(2s+1)^2} \left\langle (\mathrm{Tr}(\phi))^2 \right\rangle , \tag{4.6}$$

$$\left\langle \sum_{m=-1}^{1} |c_{1m}|^2 \right\rangle = \frac{12\pi}{s(s+1)(2s+1)^2} \left[|\mathrm{Tr}(\phi L_+)|^2 + (\mathrm{Tr}(\phi L_0))^2 \right] , \tag{4.7}$$

where $L_+ = L_1 + iL_2 \propto \hat{Y}_{11}$. The first two observables also happen to be invariant under $\mathrm{U}(n)$ transformations while the latter is only invariant under $\mathrm{SU}(2)$ transformations. These quantities can respectively be related to the average of the total power of the field, and of its power in the modes 0, and 1.

The simulations themselves are realised in the standard way using a Metropolis Monte-Carlo method with the jackknife method to evaluate the error on the calculated expected values [14]. To better control thermalisation, simulations were run starting from both hot (i.e. random) and cold (i.e. the minimum of the action) initial conditions.

[2] The usual susceptibility has no peak for finite volume because by symmetry $\langle \mathrm{Tr}(\phi) \rangle = 0$. Instead the standard finite-volume susceptibility is used where the latter term is replaced by $\langle | \mathrm{Tr}(\phi)| \rangle$ which has the right properties.

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This generates a sequence of random field configurations $\phi(t)$ with probability distribution given by e^{-S}/Z as found in the expectation values (4.2). The parameter t will be called "Monte-Carlo time", and the expectation values will then be calculated as standard averages

$$\langle h(\phi)\rangle = \frac{1}{T}\sum_{t=1}^{T} h(\phi(t))\,. \tag{4.8}$$

The update of the matrix $\phi(t)$ through one Monte-Carlo time step is done entry by entry as this is not slower than a global matrix change. With the acceptance rate defined as the number of entries modifed during a sweep over the total number of entries tested (which is all of them), the range of variation of each matrix entry is fixed adaptatively by maintaining the acceptance rate between 15% and 30%.

The key model-dependent ingredient in this method is the calculation of the variation of the action under a random variation of a matrix entry, $\delta S_{ij}(x) = S(\phi_{ij} \to \phi_{ij}+x)$ which is used to calculate the probability $\min(\mathrm{e}^{-\delta S}, 1)$ of accepting $\phi_{ij}+x$ as the matrix entry at the next Monte-Carlo timestep, and the calculation of the observables which is used in the average formula (4.8) and is always negligible in terms of the number of operations compared to $\delta S_{ij}(x)$.

The scalar field on the fuzzy sphere is "non-local" in the sense that a matrix entry ϕ_{ij} is coupled to all the other entries on its line ϕ_{ik} and column ϕ_{kj} through the interaction term of the potential $\lambda\phi^4$. As a result, the variation of the action δS requires a number of operations which grows linearly with the matrix size $2s+1$. Since each entry of the matrix must be updated from one Monte-Carlo timestep to the next, the computation time of a matrix update for the fuzzy action (3.3) grows like $\mathcal{O}(s^3)$. The quadratic part of the action only couples it to a fixed number of other entries, so that ϕ_{ij} only couples to $\phi_{i+1\,j+1}$, ϕ_{ij} and $\phi_{i-1\,j-1}$, and is therefore subdominant in this calculation.

By comparison, for a "local" action, such as that of a finite difference scalar field action on the lattice, a degree of freedom (or lattice site) is only coupled to a constant number of other degrees of freedom, i.e. independant of the total number of degrees of freedom, and the number of operations to update the field through one Monte-Carlo timestep only grows like the number of degrees of freedom $\mathcal{O}(s^2)$.

This calls for two observations. First, the only "non-local term" in the fuzzy action is actually the self-interaction term $\lambda\phi^4$. It should be noted in passing that the non-locality gets worse as the power of the self-interacting term increases, such as adding a term in $\mathrm{Tr}(\phi^6)$, in the sense that the number of operations to calculate δS_{ij} grows polynomially even faster. Second, although this model cannot be compared to a lattice model because of the UV-IR mixing, a corrected action such as those proposed in [13] which would converge towards the clasical real scalar field theory would seem to be intrinsically slower than its lattice equivalent. This however does not generalise to other field theories, particularly fermionic ones, and does not take into account the rate of convergence of the Monte-Carlo scheme itself.

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Let us now go over the various phases which arise in the simulation. In the following, as a convention when either of the two parameters λ or R is not mentionned as a variable, it is assumed to be equal to one.

4.1 The uniform-disordered phase transition

This is the phase transition observed for a ϕ^4 scalar field theory on the commutative plane. For the model considered here, it appears for small interaction parameters λ. Figure 1 shows a typical example of this phase transition. The critical point can readily be identified as the maximum of the susceptibility (4.1).

As seen on figure 1b, the observables above the critical point $r > r_c^{u-d}$ show no strong dependance on r. The power in the 0 and 1 modes is also much smaller than the total power suggesting that the random fluctuations are spread out over all the modes. This is characteristic of a disordered phase where the field is composed entirely of random fluctuations around the constant average field $\phi = 0$. This phase is found when the mass parameter r is positive or negative and "small", $-r \lesssim \lambda$. These results are completely consistent with what one would expect from a commutative scalar field theory.

The uniform phase arises when the mass parameter r is negative and "large". It appears in figure 1 below the critical point $r < r_c^{u-d}$. In this approximation, the action is completely dominated by the regions around its minima at $\phi_\pm = \pm\sqrt{-r/2\lambda}\,\mathbf{1}$, where $\mathbf{1}$ denotes the unit matrix. In this limit, it is possible to expand the action to dominant (quadratic) order around these two minima to derive the observables. This gives an effective action which is basically the sum of two delta distributions centered at each of the action

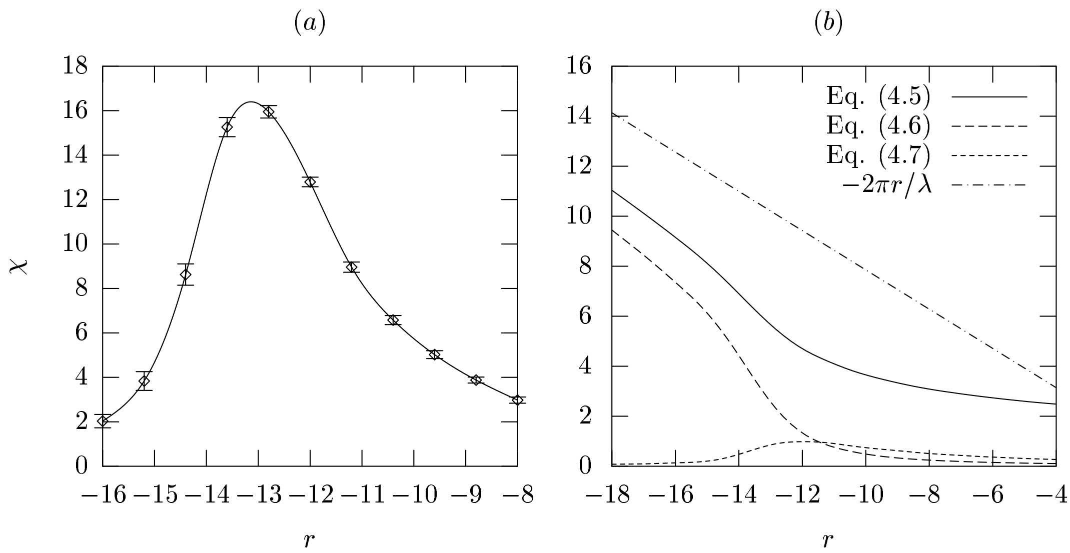

Figure 1: (a) Plot of the susceptibility from eq. (4.1), calculated on 2^{16} bins of 40 Monte-Carlo timesteps for $\lambda = 8$ and 21×21 matrices. Its maximum is evaluated at $r_c^{u-d}(\lambda = 8) = -13.2 \pm 0.6$. ($b$) Plot of the three observables described in eqs. (4.5), (4.6), (4.7) for the same parameters.

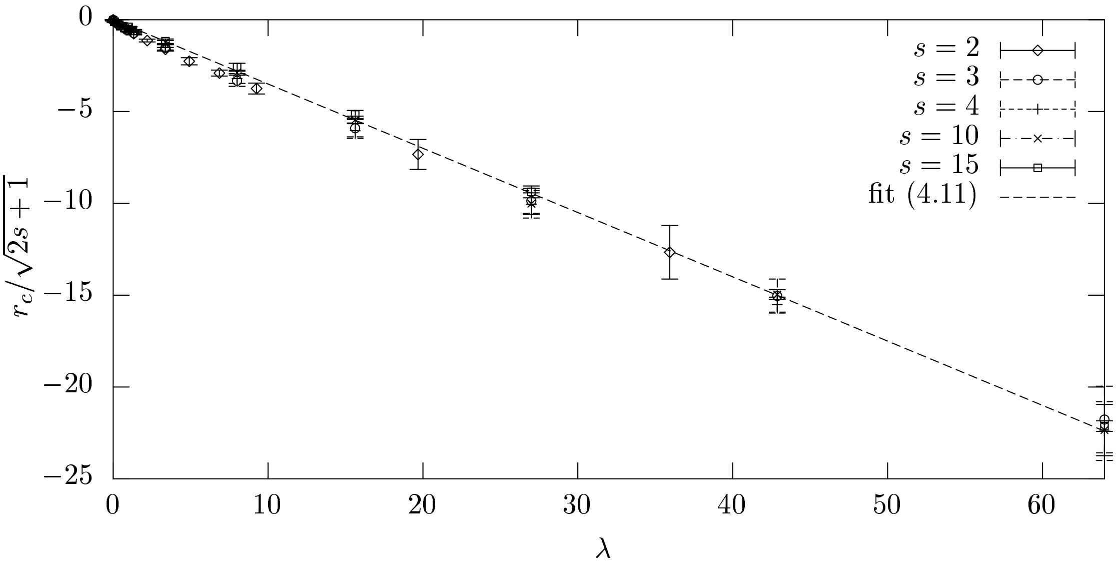

Figure 2: Scaled phase diagram for the uniform-disordered phase transition for five distinct matrix sizes $(2s+1) \times (2s+1)$ and the best fit given by (4.11).

minima $\phi_{\pm}$. In particular, as seen in figure 1*b*,

$$\left\langle \|\phi\|^2 \right\rangle \simeq \left\langle |c_{00}|^2 \right\rangle \simeq \frac{-2\pi r}{\lambda} \tag{4.9}$$

$$\left\langle \sum_m |c_{1m}|^2 \right\rangle \ll \left\langle |c_{00}|^2 \right\rangle . \tag{4.10}$$

Checking the next order of expansion shows that the power in all the non-zero modes is suppressed as $1/(-r)$.

The corresponding critical line in the phase diagram is shown in figure 2. The parameters r and λ were chosen as variables because the critical line goes through the point $(0,0)$ around which it is delicate to scale the parameters using the radius R^2. This critical line scales like the square root of the matrix size. Furthermore, a simple fit of the datas suggests that it is well approximated by

$$\frac{r_c^{u-d}(\lambda)}{\sqrt{2s+1}} \simeq -0.35\lambda\, . \tag{4.11}$$

The phase boundary line is well approximated by a linear function, but its slope seems to change slightly with the matrix size. The fit proposed above has been done on the largest matrix size data, $s = 15$, which is hopefully the best approximation to the large s limit.

It would be interesting to study in more detail this phase transition and how it compares to the disordered-uniform phase transition of the commutative scalar field theory near the origin. However, the focus of this paper is the new phase, which is found and described in the next subsection. This study is therefore left for a future paper [15].

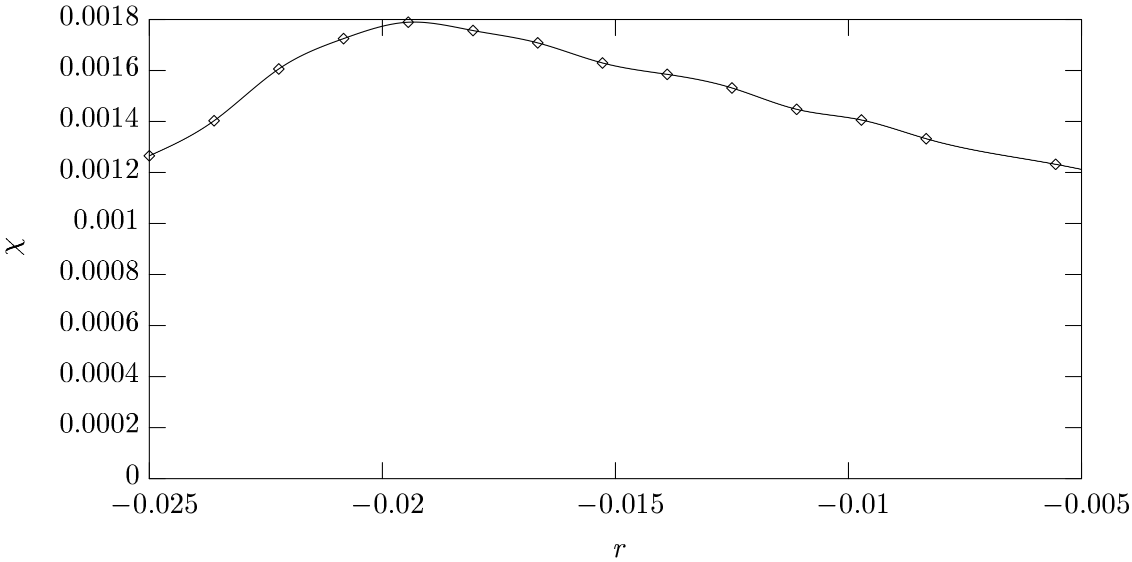

Figure 3: Plot of the susceptibility (4.1) calculated for 21×21 matrices on 2^{16} bins of 40 Monte-Carlo timesteps for $R^2 = 72^3$. Its maximum is evaluated at $r_c^{d-m}(72^3) = -0.019 \pm 0.005$.

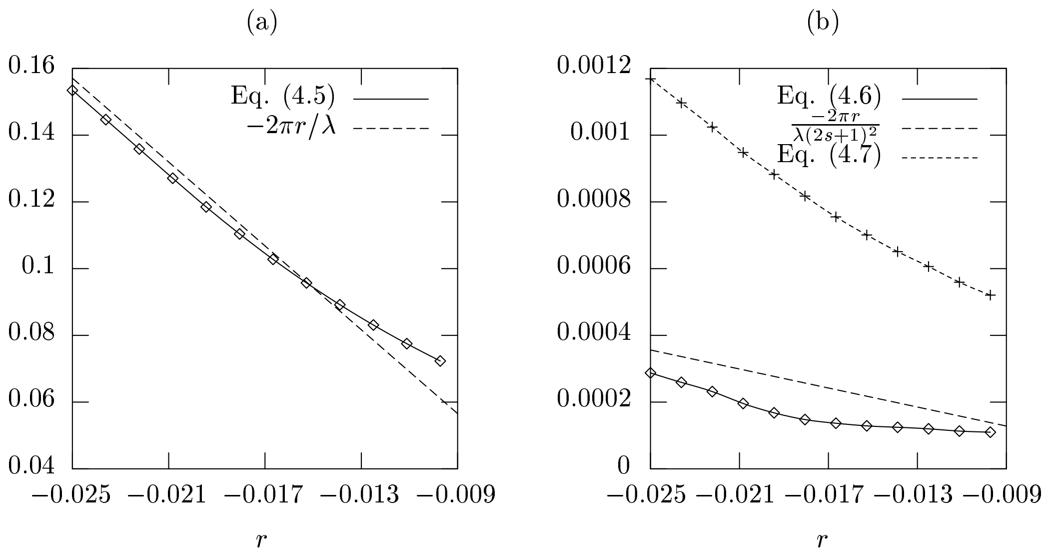

Figure 4: (a) Plots of the total power (4.5) and its proposed approximation (4.12) calculated for the same parameters as in figure 3. (b) Plots of the power in the 0 mode (4.6), its proposed approximation (4.13), and the power in the 1 mode (4.7) calculated with the same parameters.

4.2 The disordered-matrix phase transition

The simulations show the appearance of a new phase for larger radii which I will call the "matrix phase" for reasons explained in the paragraph "The pure potential model" of this subsection. Figures 3 and 4 show a typical example of this phase transition. Again the phase transition can be clearly identified as occuring at the maximum of the susceptibility.

The new phase seems characterised by

$$\left\langle \frac{1}{2s+1}\,\mathrm{Tr}(\phi^2)\right\rangle \simeq \frac{-2\pi r}{\lambda} \tag{4.12}$$

$$\langle c_{00}^2\rangle \simeq \frac{-2\pi r}{\lambda(2s+1)^2} \ll \left\langle \frac{1}{2s+1}\,\mathrm{Tr}(\phi^2)\right\rangle \tag{4.13}$$

$$\left\langle \sum_m |c_{1m}|^2\right\rangle > \langle c_{00}^2\rangle\,. \tag{4.14}$$

This suggests strongly that in this new phase the field ϕ takes the form $\phi \sim \pm\sqrt{-r/2\lambda}\times U^\dagger(\mathbf{1}_s \oplus -\mathbf{1}_{s+1})U$ with $U \in \mathrm{U}(2s+1)$ and $\mathbf{1}_n$ the $n\times n$ unit matrix, which is a minimum of the potential (3.4).

Further examination of the raw data of a Monte-Carlo run with cold initial conditions shows that during thermalisation $\mathrm{Tr}(\phi(t))$ (t being the Monte-Carlo time) goes through a series of plateaus at $(2l+1)\sqrt{-r/2\lambda}$, $|l| \le s$ before settling down at its equilibrium value corresponding to $l = 0$, which is consitent with the value of $< c_{00}^2 >$ given in eq. (4.13). This seems to confirm that the minima of the potential part of the action given by $\sqrt{-r/2\lambda}\,U^\dagger(\mathbf{1}_l \oplus -\mathbf{1}_{2s+1-l})U$ with $U \in \mathrm{U}(2s+1)$ are local minima of the action.

Normally in the uniform phase, the minimum corresponding to $l \in \{0, 2s+1\}$ is automatically selected by virtue of also minimizing the kinetic part of the action. The fact that in the matrix phase it is not suggests that the kinetic term might be negligible. To test this assertion, the field theory described by the action $V(\phi)$ from eq. (3.4) has been studied and compared to the results for the fuzzy scalar field.

The pure potential model. This model is actually much simpler than the fuzzy scalar field action by virtue of being invariant under $\mathrm{U}(2s+1)$ transformations of the field $\phi \to U^\dagger\phi U$, $U \in \mathrm{U}(2s+1)$.

Thus, considering only observables which are also invariant under these transformations, such as those defined in eqs. (4.1), (4.5), (4.6), the degrees of freedom associated with this invariance can be extracted in the form $\phi = U^\dagger \mathrm{Diag}(x_1,\ldots,x_{2s+1})U$, which has a jacobian given by the square of the Vandermonde determinant where the Vandemonde determinant is given by $\Delta(x_1,\ldots,x_{2s+1}) = \prod_{i<j}(x_i - x_j)$, and integrated out to get an effective action given by

$$V_{\mathrm{eff}}(x_1,\ldots,x_{2s+1}) = \frac{4\pi R^2}{2s+1}\sum_{i=1}^{2s+1}(rx_i^2 + \lambda x_i^4) - \sum_{1\le i<j\le 2s+1}\ln[(x_j - x_i)^2]\,. \tag{4.15}$$

Furthermore, in the case of the observables defined in eqs. (4.1), (4.5), (4.6), generically called $F(r,\lambda,R)$, which are quadratic in the field, a simple change of variable $y_i = R^{1/2}\lambda^{1/4}$, shows that these averages effectively depend only on one parameter

$$R\sqrt{\lambda}\,F(r,\lambda,R) = F\left(\frac{r}{(R\sqrt{\lambda})},1,1\right). \tag{4.16}$$

This effective action (4.15) can be easily studied by Monte-Carlo simulations since it has a lot fewer degrees of freedom. Figure 5 supports the assertion that around the

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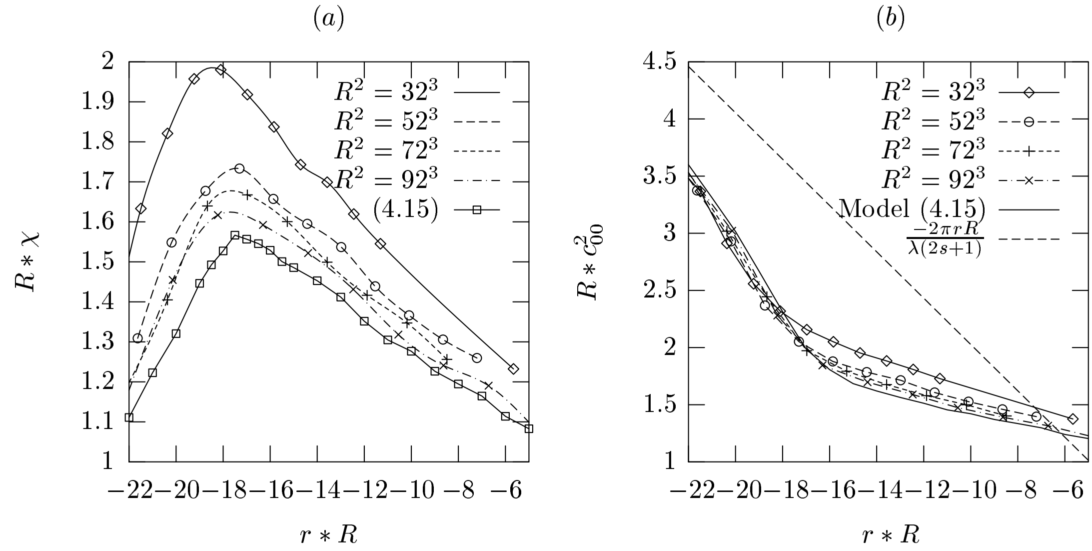

Figure 5: (a) Plot of the susceptibility for 31×31 matrices, in units such that they do not depend on the radius in the case of the pure potential action as shown in eq. (4.16). Note that the critical point for the pure potential model is at $r_c^{d-m}R = -17.5 \pm 0.5$. (b) Plot of $< c_{00}^2 >$ and its approximation (4.13) for the same parameters. In both figures (a) and (b), the observables calculated from the fuzzy scalar field action seem to converge monotonically with R^2 toward the pure potential observables.

phase transition considered here, the kinetic term of the fuzzy scalar field model becomes negligible in the limit when the radius R^2 tends to infinity. This is why the new phase was called a "matrix phase". In this phase, the kinetic term which carries the geometrical content of the fuzzy sphere vanishes. It is likely that the "striped phase" which has been found for the ϕ^4 theory on the non-commutative lattice [8] arises for the same reason and must therefore converge to the same matrix phase. In particular, for the same parameters and deep in the matrix phase, the values of $\mathrm{U}(2s+1)$ invariant observables, such as $\langle \mathrm{Tr}(\phi) \rangle$, $\langle \mathrm{Tr}^2(\phi) \rangle$, or $\langle \mathrm{Tr}(\phi^2) \rangle$, should be the same on the non-commutative lattice as on the fuzzy sphere.

The matrix phase. Eq. (4.12) suggests that in the matrix phase the eigenvalues of the field settle in a minimum of the potential. This suggests that only the neighbourhood of the minima of the potential contribute to the expectation values (4.2).

The minima of the potential are $2s + 1$ disjoint orbits of the form

$$O_n = \left\{ \sqrt{\frac{-r}{2\lambda}} U^\dagger (\mathbf{1}_n \oplus -\mathbf{1}_{2s+1-n}) U | U \in \mathrm{U}(2s+1)/[\mathrm{U}(n) \times \mathrm{U}(2s+1-n)] \right\}, \qquad (4.17)$$

where $n \leq s + 1/2$ and $\mathbf{1}_n$ are the $n \times n$ unit matrices. These orbits are isomorphic to Grassmanians $\mathrm{G}_n^{2s+1} \cong \mathrm{U}(2s+1)/[\mathrm{U}(n) \times \mathrm{U}(2s+1-n)]$. When the kinetic term is negligible, the field distribution will be dominated by the orbit with the largest phase space volume which corresponds to n the integer value out of s and $s + 1/2$.

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By comparison, the kinetic term which was neglected will select the diagonal orbits, $n \in \{0, 2s+1\}$, which have the lowest phase space volume. Thus, the matrix phase can be interpreted as a phase where the kinetic term is negligible with respect to the volume of the largest orbit $\mathcal{O}_s$.

This reasoning can be verified more rigorously, by looking carefully at the probability measure

$$\mathrm{d}\mu(\phi) = \frac{\mathrm{e}^{-S(\phi)}}{Z} d\phi \tag{4.18}$$

asociated with the field theory. The simulations indicated clearly that only the fields near the minimum of the potential contribute to the distribution. Therefore, expanding ϕ around these minima in the form

$$\phi = x_0\, U^\dagger \mathrm{Diag}(\epsilon_j + x_j)\, U\,, \tag{4.19}$$

with $U \in \mathrm{U}(2s+1)$, $x_0 = \sqrt{-r/2\lambda}$ the minimum of the potential, and $\epsilon_j \in \{-1, +1\}$, the probability distribution becomes

$$\begin{aligned}\mathrm{d}\mu(\phi) \approx \frac{1}{Z}\sum_{\epsilon_j} \delta(\phi - x_0\, U^\dagger \mathrm{Diag}(\epsilon_j + x_j)\, U)\, \mathrm{e}^{-4\pi (rR)^2 \mathbf{x}\cdot\mathbf{x}/[\lambda(2s+1)] - x_0^2 K(U,\epsilon_j,x_j)} \times \\ \times h V^2(\epsilon_j + x_j)\, \mathrm{d}U\, \mathrm{d}^{2s+1}\mathbf{x}\,,\end{aligned} \tag{4.20}$$

where

$$K(U, \epsilon_j, x_j) = \frac{4\pi}{2s+1}\, \mathrm{Tr}[(U^\dagger \mathrm{Diag}(\epsilon_j + x_j)\, U)\mathcal{L}^2(U^\dagger \mathrm{Diag}(\epsilon_j + x_j)\, U)] \tag{4.21}$$

is the kinetic term for the field expanded acording to eq. (4.19) and Z is the partition function whose value changes as needed to normalise the probability distribution it is part of. It is possible to simplify this expression even further by ordering the eigenvalues x_i since such permutations are also $\mathrm{U}(2s{+}1)$ transformation. Thus, dropping all subdominant terms, and scaling $\mathbf{x}$,

$$\begin{aligned}\mathrm{d}\mu(\phi) \approx \frac{1}{Z}\sum_{i=0}^{2s+1} \binom{2s+1}{i} \delta\left(\phi - x_0\, U^\dagger \mathrm{Diag}\left(\mathrm{sign}\left(i - j + \frac{1}{2}\right) + \sqrt{\frac{(2s{+}1)\lambda}{4\pi}}\frac{y_j}{rR}\right) U\right) \times \\ \times \left(\mathrm{e}^{-x_0^2 K(U, \mathrm{sign}(i-j+1/2), 0)} \mathrm{d}U\right) \left(\left(\frac{16\pi r^2 R^2}{\lambda(2s{+}1)}\right)^{i(2s+1-i)} \mathrm{e}^{-\mathbf{y}\cdot\mathbf{y}} V^2(y_{j\le i}) V^2(y_{j>i})\, \mathrm{d}\mathbf{y}\right),\end{aligned} \tag{4.22}$$

where each term of the sum corresponds to an integration of the action in the vicinity of the $\mathrm{U}(2s+1)$ orbit of $\mathbf{1}_i \oplus -\mathbf{1}_{2s+1-i} = \mathrm{sign}(i - j + 1/2)$. In this sum, the "kinetic" term (containing $\mathrm{d}U$) decreases with i while the "potential" term (containing $\mathrm{d}\mathbf{x}$) increases with i. Thus, the matrix phase corresponds to the case when the potential term dominates, whereas the uniform phase would correspond to one where the kinetic term dominates.

Note that, by reabsorbing the Vandermonde determinants into matrix integrations, the potential term can be rewritten as a gaussian measure

$$\mathrm{e}^{-\mathbf{x}\cdot\mathbf{x}} V^2(x_{j\le i}) V^2(x_{j>i})\, \mathrm{d}\mathbf{x} = \left(\int_{\mathrm{SU}(i)} \frac{\mathrm{d}U_1}{\mathcal{V}(\mathrm{SU}(i))}\right) V^2(x_{j\le i}) \left(\prod_{j=1}^{i} \mathrm{d}x_j\right) \mathrm{e}^{-\sum_{j=1}^{i} x_j^2} \times \tag{4.23}$$

$$\times \left(\int_{\mathrm{SU}(2s-i)} \frac{\mathrm{d}U_2}{\mathcal{V}(\mathrm{SU}(2s-i))} \right) \left(\prod_{j=i+1}^{2s+1-i} \mathrm{d}x_j \right) \mathrm{e}^{-\sum_{j=i+1}^{2s+1-i} x_j^2}$$

$$\propto \mathcal{V}(\mathrm{Gr}_{i,2s+1})\, \delta(\psi_1 - U_1^\dagger \mathrm{Diag}(x_{j\leq i})\, U_1) \times \tag{4.24}$$

$$\times \delta(\psi_2 - U_2^\dagger \mathrm{Diag}(x_{j>i})\, U_2) \int_{\mathrm{SU}(i)\times \mathrm{SU}(2s-i)} \mathrm{d}\psi_1 \mathrm{d}\psi_2\, \mathrm{e}^{-\mathrm{Tr}(\psi_1^2+\psi_2^2)},$$

where $\mathcal{V}$ denotes the volume of a space. This gaussian integral can be calculated exactly for most expectation values. Furthermore, it makes explicit the volumes of the orbits $\mathcal{O}_i \sim \mathrm{Gr}_{i,2s+1}$ which weights these integrals. Not surprisingly now, the power $i(n-i)$ which appears in the potential term is precisely the dimension of this Grassmanian.

Note also that for the purely potential action, the terms $i = s$ or $i = s \pm 1/2$, whichever are integer, are always dominant over the others, and so in this limit (which excludes the disordered phase because of the approximation (4.19)) only the matrix phase can arise.

This extra phase does not appear for the scalar field simulated on the lattice despite the fact that the action has the same superficial properties. The potential has a large subset of minima given by $\phi_{ij} = \pm x_0$ at each lattice site. The minimum with the largest phase space is the one with half their sites with value $+x_0$ and half with the opposite value $-x_0$ which has a degeneracy of $\binom{N^2}{N^2/2}$ where N^2 is the number of lattice points, whereas the minimum of the action is the one with a uniform value at each site which has just degeneracy two. However, on the lattice the kinetic term simply cannot be neglected because it represents the only, and thus *dominant*, coupling between the degrees of freedom at each lattice point. Thus, it must lift the degeneracy of the minimum of the potential in favour of the usual uniform phase. By contrast, on the fuzzy sphere, the non-locality of the potential implies that the coupling of the matrix degrees of freedom is ensured by both the kinetic and potential term.

The equilibrium configuration. As described above, in the matrix phase, the field will settle in the vicinity of the orbit $\mathcal{O}_s$. However, although this orbit is degenerate with respect to the potential term (3.4), it is not with respect to the full action because the kinetic term will lift this degeneracy.

Taking into account the kinetic term, the most probable configuration of the field will be given by the minimum of the kinetic term (or equivalently of the action) on the orbit $\mathcal{O}_s$. The theory described by the scalar action restricted to the orbits $\mathcal{O}_s$ has been studied in detail in [16]. The configurations found to minimise the action on these orbits $\mathcal{O}_i$ were found to be of the form $x_0 W^\dagger\, (\mathbf{1}_i \oplus -\mathbf{1}_{2s+1-i})\, W$, with $W \in SU_s(2)$ the $2s+1$ dimensional representation of SU(2). So, the equilibrium configuration in the matric phase is found to be of the form $\pm x_0 W^\dagger\, (\mathbf{1}_s \oplus -\mathbf{1}_{s+1}) W$, with $W \in SU_s(2)$.

Applying the mapping (2.6), it is possible to extrapolate from there the form this field will take in the commutative limit. Since this mapping preserves the action of SU(2), the corresponding set of fields will be given by $\pm x_0 W^\dagger\, \mathcal{M}_s(\mathbf{1}_s \oplus -\mathbf{1}_{s+1})\, W$, where $W \in \mathrm{SO}(3)$ is now a simple global rotation. Figure 6 shows the function with azimuthal symmetry

$$f_s(\theta) = \mathcal{M}_s(-\mathbf{1}_s \oplus \mathbf{1}_{s+1})\,, \tag{4.25}$$

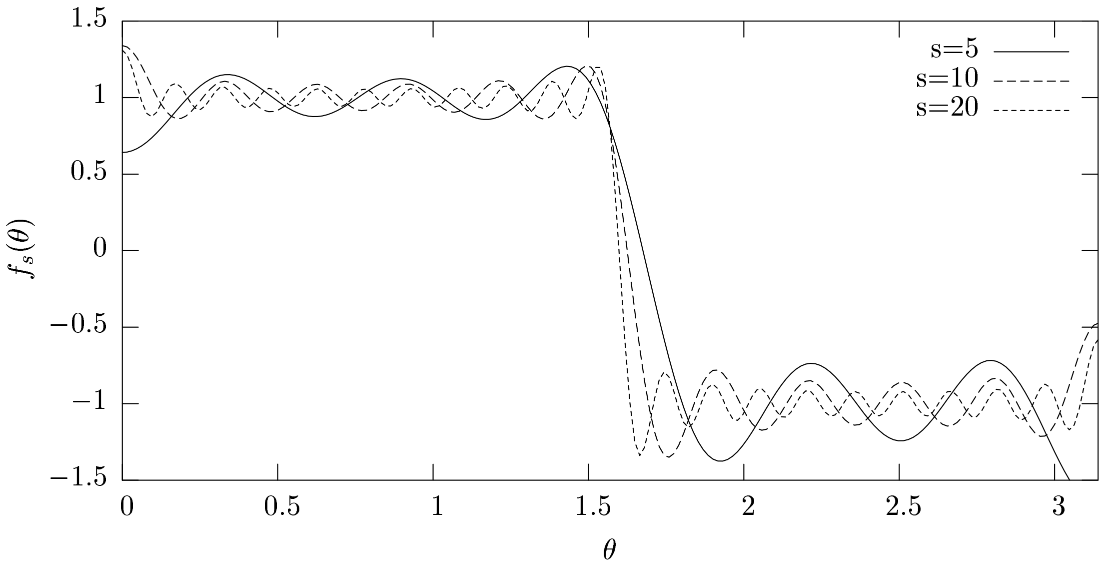

Figure 6: The function $f_s(\theta)$ defined in eq. (4.25) for various values of the matrix size $2s+1$. Note how the functions tend to a sign function.

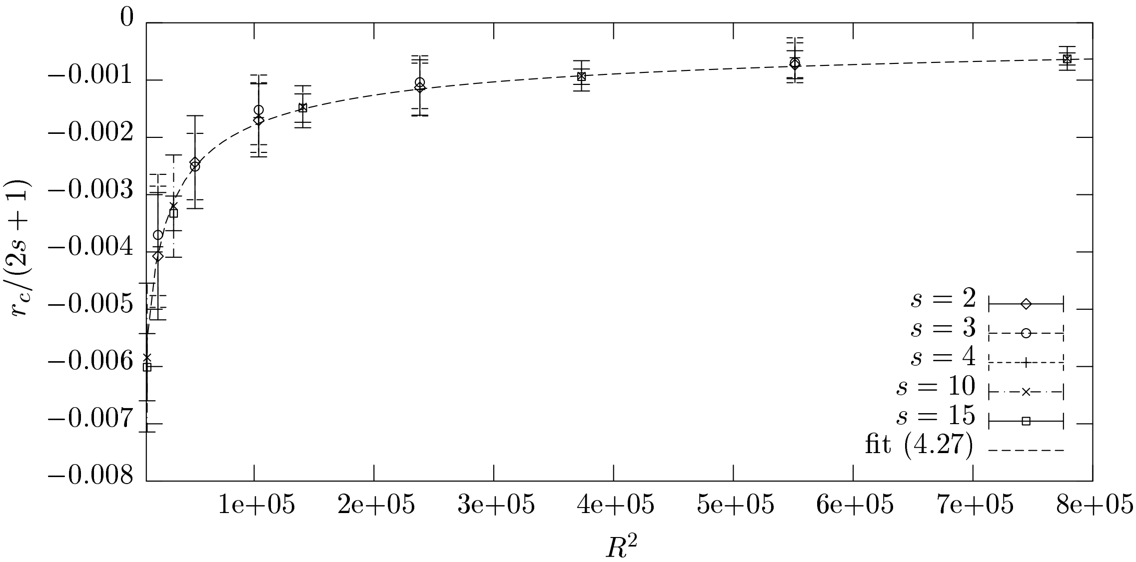

Figure 7: Scaled phase diagram for the disordered-matrix phase transition for five distinct matrix sizes $(2s+1) \times (2s+1)$ and the asymptotic limit (4.27) which fits the curves well for $R^2 \geq 30$.

with θ the zenith angle. Thus the matrix phase on the fuzzy sphere has a commutative limit similar to the one observed on the torus [8]. It also suggests *a posteriori* that the name of "striped phase" might also be appropriate for this phase.

The critical line. The critical line for this phase transition scales linearly with the matrix size, i.e. like s. Figure 7 shows the critical lines. For large radii, a fit can easily be deduced algebraically from the collapse of the susceptibility curves and the critical point for the pure potential model shown in figure 5a, and the scaling of the susceptibility for

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the pure potential model shown in (4.16). Indeed, for $s = 15$, and $R \to +\infty$

$$r_c^{d-m}(R^2) \sim \frac{-17.5}{R}\,, \tag{4.26}$$

thus, putting back in the linear scaling found for the critical line,

$$\frac{r_c^{d-m}(R^2)}{2s+1} \sim \frac{-17.5}{31R} \simeq -0.56\,(R^2)^{-1/2}\,, \tag{4.27}$$

which fits the critical line quite well as shown in figure 7.

4.3 The matrix-uniform phase transition

This transition is difficult to observe numerically because of thermalisation problems. To switch from the matrix phase to the disordered phase requires a large change in the field between two local minima of the action, which has numerically vanishing low probability of happening. In fact, starting from hot initial conditions (i.e. random), this transition never appears and the field stays trapped in the matrix phase. This is quite understandable since we have seen in eq. (4.22) that the phase space asociated with the matrix phase is so much bigger than the one associated with the uniform phase.

Conversely, with cold initial conditions (i.e. the field is a minimum of the action or equivalently in the uniform phase), when $-r$ gets large enough, the field stays trapped near this locally stable configuration, and a matrix-uniform phase appears. However, since hot initial conditions give a different result, the observed transition just pinpoints the parameter r when the inverse of the probability of tunneling to the matrix phase becomes larger than the thermalisation time of the Monte-Carlo simulation.

This is why the uniform-disordered phase diagram 2 is truncated for $R^2 > 64$. Beyond this point, the three phases uniform, matrix, and disordered, start mixing and runs with the cold initial conditions show the emergence of an uniform phase while the hot do not.

Still, it is possible to guess a few things about the large $-r$ region. As discussed in subsection 4.2, the term $i = s$ in the sum of eq. (4.22) corresponds to the matrix phase while the term $i = 0$ corresponds to the uniform phase. When R and s are fixed and r increases, the potential term grows polynomially whereas the kinetic one is suppressed exponentially. Thus, the uniform phase must dominate when $-r$ becomes large enough, and there *is* a matrix-uniform phase transition.

Furthermore, when the kinetic term start dominating over the potential term in (4.22), its exponential dependence should ensure that the term $i = 0$ in the sum quickly becomes dominant. Thus, it is unlikely that there be other intermediate phases, associated with other minima of orbits $\mathcal{O}_i$ $i \notin \{0, s\}$, between the matrix and uniform phases.

Note that the pure potential model is no help here as it can only have two phases: the disordered and matrix phases. This is not surprising as the kinetic term is a key component in this region of the phase diagram.

A systematic analysis of this phase transition, and by extension of the triple point where the three phases coexist, will be presented in the future paper [15] already mentionned at the end of section 4.1.

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5. Conclusions

In conclusion, the scalar field action on the fuzzy sphere shows the emergence of a new phase and exhibits a new aspect of the phenomenon commonly called UV-IR mixing. The critical line for the uniform-disordered phase transition is identified and scales like the square root of the matrix dimension. The critical line for the matrix-disordered phase transition is also identified and found to scale linearly with the matrix size and grows linearly for large sphere radii. Finally, the existence of a matrix-uniform phase transition is ascertained algebraically, but could not be identified numerically due to thermalisation problems. We also conjecture that there is no other intermediate phase between the matrix and uniform phases.

The matrix phase was also studied in detail, showing that its emergence over the uniform phase is linked to the dominance of the phase space volume of the largest orbit minimising the potential, over the kinetic term. As a result, the matrix phase must be well approximated for large sphere radii by a pure potential model, i.e. one with no kinetic term. This was confirmed numerically.

Since, for a scalar field theory, the geometry of a fuzzy space is determined by the choice of its laplacian or equivalently of the kinetic term of the scalar field action, the matrix phase appears to be largely independant of the geometry. Thus, if it appears in other scalar matrix models, it must have similar properties to those shown here, independantly of the geometry or dimension of the commutative limiting space. In particular, it should be possible to verify this by comparing the results found for the fuzzy sphere and the non-commutative lattice [8].

Finally, the equilibrium configuration was identified indirectly and shown to have a "striped" structure similar to what has been observed on the non-commutative lattice.

This is of course a verification that the naive scalar field action studied in this paper cannot be used as an approximation of the scalar field theory on the usual sphere. For the latter, one needs to look at a more complicated action such as the one proposed in [13] which adds an extra damping term proportional to $\mathrm{Tr}((\mathcal{L}^2\phi)^2)$ to the action to suppress UV-IR mixing. This is consistent with the analysis presented here as such a term will reinforce the influence of the kinetic term and thus suppress the matrix phase.

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